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Some properties of Hermite based Appell matrix polynomials

Subuhi Khan[a], Ahmed Ali Al-Gonah[b] and Ghazala Yasmin [c,*]

[a,c]Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
[b]Department of Mathematics, Aden University, Aden, Yemen
* Corresponding author

E-mail: subuhi2006@gmail.com[a], gonah1977@yahoo.com[b], ghazala30@gmail.com[c]

Abstract

In this paper, the Hermite based Appell matrix polynomials are introduced by using certain operational methods. Some properties of these polynomials are considered. Further, some results involving the 2D Appell polynomials are established, which are proved to be useful for the derivation of results involving the Hermite based Appell matrix polynomials.

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1 Introduction

An important generalization of special functions is special matrix functions. The study of special matrix polynomials is important due to their applications in certain areas of statistics, physics and engineering. Matrix analogues of Laguerre, Hermite and Legendre differential equations and the corresponding polynomial families are studied in [11-13]. The Hermite matrix polynomials and their extensions and generalizations have been introduced and studied for matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues are all situated in the right open half-plane, see for example [11, 14, 19, 20, 22].

We review the definitions and the concepts related to the Hermite matrix polynomials.

Throughout the paper unless otherwise stated, we assume that A is a positive stable matrix in $\mathbb{C}^{N\times N}$, that is, A satisfied the following condition:

$$Re(\mu) > 0, \quad \text{for all } \mu \in \sigma(A), \tag{1.1}$$

where $\sigma(A)$ denotes the set of all the eigenvalues of A.

If D_0 is the complex plane cut along the negative real axis and $log(z)$ denotes the principal logarithm of z, then $z^{1/2}$ represents $\exp(\frac{1}{2}log(z))$. If matrix $A \in \mathbb{C}^{N\times N}$ with $\sigma(A) \subset D_0$, then $A^{1/2} = \sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus [9] acting on the matrix A. We consider the 2-index 2-variable Hermite matrix polynomials (2I2VHMP) $H_{n,m}(x,y,A)$, which are defined by the series [22; p.689]

$$H_{n,m}(x,y,A) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k y^k (x\sqrt{mA})^{n-mk}}{(n-mk)!k!} \quad (n \geq 0) \tag{1.2}$$

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and specified by the generating function

$$\exp(xt\sqrt{mA} - yt^m I) = \sum_{n=0}^{\infty} H_{n,m}(x, y, A)\frac{t^n}{n!}, \tag{1.3}$$

where I is the unit matrix in $\mathbb{C}^{N\times N}$.

We recall that according to the monomiality principle [4, 23] a polynomial set $\{p_n(x)\}$ ($n \in \mathbb{N}, x \in \mathbb{C}$) is quasi-monomial if there exist two operators $\hat{M}$ and $\hat{P}$, called respectively the multiplicative and derivative operators, which when acting on the polynomials $p_n(x)$ yield:

$$\hat{M}\{p_n(x)\} = p_{n+1}(x), \tag{1.4a}$$

$$\hat{P}\{p_n(x)\} = np_{n-1}(x). \tag{1.4b}$$

The operators $\hat{P}$ and $\hat{M}$ satisfy the commutation relation

$$[\hat{P}, \hat{M}] = \hat{1} \tag{1.5}$$

and thus display a Weyl group structure. If the considered polynomials set $\{p_n(x)\}$ is quasi-monomial, its properties can be deduced from those of the $\hat{M}$ and $\hat{P}$ operators. If the operators $\hat{M}$ and $\hat{P}$ have a differential realization, then the polynomials $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}\{p_n(x)\} = np_n(x). \tag{1.6}$$

Assuming here and in the following $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as:

$$p_n(x) = \hat{M}^n\{1\} \tag{1.7}$$

and consequently the generating function of $p_n(x)$ can be cast in the form

$$\exp(t\hat{M})\{1\} = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}. \tag{1.8}$$

We note that the 2I2VHMP $H_{n,m}(x, y, A)$ are quasi-monomial under the action of the operators [21; p.43]

$$\hat{M}_H := x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\ , \tag{1.9a}$$

$$\hat{P}_H := \frac{1}{\sqrt{mA}}\ \frac{\partial}{\partial x}\ . \tag{1.9b}$$

The 2I2VHMP $H_{n,m}(x, y, A)$ are also defined through the operational rule [22; p.699]

$$H_{n,m}(x, y, A) = \exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)\left\{(x\sqrt{mA})^n\right\}. \tag{1.10}$$

Next, we recall that the 3-index 3-variable Hermite matrix polynomials (3I3VHMP) $H_n^{(m,s)}(x,y,z;A)$ are defined by the series [19]

$$H_n^{(m,s)}(x,y,z;A) = n!\sum_{k=0}^{[\frac{n}{s}]}\sum_{r=0}^{[\frac{n-sk}{m}]}\frac{(-1)^r z^k y^r (x\sqrt{mA})^{n-sk-mr}}{k!r!(n-sk-mr)!} \quad (n \geq 0) \tag{1.11}$$

and specified by the generating function

$$\exp(xt\sqrt{mA} - yt^m I + zt^s I) = \sum_{n=0}^{\infty} H_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!}, \tag{1.12}$$

Note that, for $y = 0$, the 3I3VHMP $H_n^{(m,s)}(x,y,z;A)$ reduce to the 3-index 2-variable Hermite matrix polynomials (3I2VHMP) $H_{n,m}^{(s)}(x,z;A)$ [19] defined by

$$\exp(xt\sqrt{mA} + zt^s I) = \sum_{n=0}^{\infty} H_{n,m}^{(s)}(x,z;A)\frac{t^n}{n!}, \tag{1.13}$$

which for $s = m$ and $z \to -z$, reduces to the 2I2VHMP $H_{n,m}(x,z,A)$, i.e., we have

$$H_{n,m}^{(m)}(x,-z;A) = H_{n,m}(x,z,A). \tag{1.14}$$

Also, the 3I2VHMP $H_{n,m}^{(s)}(x,y;A)$ are linked to the Gould-Hopper polynomials (GHP) $H_n^{(s)}(x,y)$ [10] by the following relation:

$$H_{n,m}^{(s)}(x,y;A) = H_n^{(s)}(x\sqrt{mA},y), \tag{1.15}$$

where $H_n^{(s)}(x,y)$ are defined by the generating function

$$\exp(xt + yt^s) = \sum_{n=0}^{\infty} H_n^{(s)}(x,y)\frac{t^n}{n!}. \tag{1.16}$$

The 3I3VHMP $H_n^{(m,s)}(x,y,z;A)$ are also defined by the following operational rule:

$$H_n^{(m,s)}(x,y,z;A) = \exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)\left\{H_{n,m}^{(s)}(x,z;A)\right\}, \tag{1.17}$$

which by using relation (1.15) gives the following equivalent operational representation:

$$H_n^{(m,s)}(x,y,z;A) = \exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)\left\{H_n^{(s)}(x\sqrt{mA},z\right\}. \tag{1.18}$$

Further, we recall that the 2D Appell polynomials $R_n^{(s)}(x,y)$ are defined by means of the generating function [2; p.835] (see also [3; p.417])

$$A(t)\exp(xt + yt^s) = \sum_{n=0}^{\infty} R_n^{(s)}(x,y)\frac{t^n}{n!}, \tag{1.19}$$

where $A(t)$ has (at least the formal) expansion

$$A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!} \quad (A_0 \neq 0) \tag{1.20}$$

and A_n are the Appell numbers.

For $y = 0$, the 2D Appell polynomials $R_n^{(s)}(x, y)$ reduce to the Appell polynomials $A_n(x)$ [1], i.e, we have

$$R_n^{(s)}(x, 0) = A_n(x), \tag{1.21}$$

where $A_n(x)$ are defined by the generating function

$$A(t) \exp(xt) = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}. \tag{1.22}$$

The explicit forms of the 2D Appell polynomials $R_n^{(s)}(x, y)$ in terms of the GHP $H_n^{(s)}(x, y)$ and vice-versa are given as [2, p.836] (see also [3]):

$$R_n^{(s)}(x, y) = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}\ H_k^{(s)}(x, y), \tag{1.23}$$

$$H_n^{(s)}(x, y) = \sum_{k=0}^{n} \binom{n}{k} Q_{n-k}\ R_k^{(s)}(x, y), \tag{1.24}$$

where Q_k are the coefficients of the Taylor expansion in a neighborhood of the origin of the reciprocal $1/A(t)$. Also, in view of generating functions (1.19), (1.22) and (1.16), we have

$$R_n^{(s)}(x + z, y) = \sum_{k=0}^{n} \binom{n}{k} R_k^{(s)}(x, y)\ z^{n-k}, \tag{1.25}$$

$$R_n^{(s)}(x + z, y) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(s)}(x, y)\ A_{n-k}(z), \tag{1.26}$$

$$R_n^{(s)}(x + z, y + w) = \sum_{k=0}^{n} \binom{n}{k} R_k^{(s)}(x, y)\ H_{n-k}^{(s)}(z, w). \tag{1.27}$$

We consider some members of the 2D Appell polynomial family $R_n^{(s)}(x, y)$ by taking suitable values of the function $A(t)$.

Taking $A(t) = \frac{t}{(e^t - 1)}$ in equation (1.19), we have [3]

$$\frac{t}{(e^t - 1)} \exp(xt + yt^s) = \sum_{n=0}^{\infty} B_n^{(s)}(x, y) \frac{t^n}{n!}, \tag{1.28}$$

where $B_n^{(s)}(x, y)$ denotes the 2D Bernoulli polynomials [3].

Next, taking $A(t) = \frac{2}{(e^t+1)}$ and $A(t) = \frac{2t}{(e^t+1)}$ in equation (1.19) and denoting the 2D Euler and 2D Genocchi polynomials by $E_n^{(s)}(x,y)$ and $G_n^{(s)}(x,y)$ respectively, we have

$$\frac{2}{(e^t+1)} \exp(xt + yt^s) = \sum_{n=0}^{\infty} E_n^{(s)}(x,y)\frac{t^n}{n!}, \tag{1.29}$$

$$\frac{2t}{(e^t+1)} \exp(xt + yt^s) = \sum_{n=0}^{\infty} G_n^{(s)}(x,y)\frac{t^n}{n!}. \tag{1.30}$$

Operational identities are useful for the algebraic decomposition of exponential operators [16]. Recently Srivastava et al. [5,7,8,15,17,18] has established some results for the sheffer and Appell polynomials and also for the new classes of mixed special polynomials related to these polynomials by employing certain operational methods. This paper is an attempt to further stress the importance of operational methods in introducing new families of special matrix polynomials. In this paper, the Hermite based Appell matrix polynomials associated with 2I2VHMP are introduced and their properties are established.

2 Hermite-Appell matrix polynomials

We introduce the Hermite-Appell matrix polynomials (HAMP) by means of the generating function. Denoting the HAMP by ${}_HR_n^{(m,s)}(x,y,z;A)$, we derive the following generating function for these polynomials:

$$A(t)\ \exp(xt\sqrt{mA} - yt^m I + zt^s I) = \sum_{n=0}^{\infty} {}_HR_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!}, \tag{2.1}$$

where m, s are both positive integers.

In order to obtain generating function (2.1), we replace x in equation (1.19) by the multiplicative operator $\hat{M}_H$ given in equation (1.9a) of the 2I2VHMP $H_{n,m}(x,y,A)$ and y by z, so that we have

$$A(t)\ \exp\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)t + zt^s\right) = \sum_{n=0}^{\infty} {}_HR_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!}. \tag{2.2}$$

Now, decoupling the exponential operator in the l.h.s. of the above equation by using the Weyl identity [6]

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{\frac{-k}{2}} \qquad \left([\hat{A},\hat{B}] = k,\ k\in\mathbb{C}\right), \tag{2.3}$$

we find

$$A(t)\ \exp(zt^s)\ \exp\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)t\right) = \sum_{n=0}^{\infty} {}_HR_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!}, \tag{2.4}$$

which on making use of the Crofton-type identity [6]

$$f\left(x + m\lambda\frac{d^{m-1}}{dx^{m-1}}\right)\{1\} = \exp\left(\lambda\frac{d^m}{dx^m}\right)\Big\{f(x)\Big\}, \tag{2.5}$$

to decouple the second exponential in the l.h.s. gives the generating function (2.1).

Next, we proceed to find the series definition of the HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$. Expanding $A(t)$ in equation (2.1) by using equation (1.20) and then using equation (1.12) in the l.h.s. of the resultant equation, we find (after equating the coefficients of like powers of t)

$$ {}_HR_n^{(m,s)}(x,y,z;A) = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}\ H_k^{(m,s)}(x,y,z;A), \tag{2.6}$$

which in view of equation (1.11) gives the following series definition of the HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$:

$$ {}_HR_n^{(m,s)}(x,y,z;A) = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}\ k! \sum_{p=0}^{[\frac{k}{s}]} \sum_{r=0}^{[\frac{k-sp}{m}]} \frac{(-1)^r z^p y^r (x\sqrt{mA})^{k-sp-mr}}{p!r!(k-sp-mr)!} \quad (k \geq 0) \tag{2.7}$$

Differentiating equation (2.1) partially with respect to x, y and z, we get the following matrix differential recurrences relations satisfied by ${}_HR_n^{(m,s)}(x,y,z;A)$:

$$ \frac{\partial}{\partial x}\ {}_HR_n^{(m,s)}(x,y,z;A) = n\sqrt{mA}\ {}_HR_{n-1}^{(m,s)}(x,y,z;A) \quad (n \geq 1)\ , \tag{2.8}$$

$$ \frac{\partial}{\partial y}\ {}_HR_n^{(m,s)}(x,y,z;A) = -\frac{n!}{(n-m)!}\ {}_HR_{n-m}^{(m,s)}(x,y,z;A) \quad (n \geq m)\ , \tag{2.9}$$

$$ \frac{\partial}{\partial z}\ {}_HR_n^{(m,s)}(x,y,z;A) = \frac{n!}{(n-s)!}\ {}_HR_{n-s}^{(m,s)}(x,y,z;A) \quad (n \geq s)\ . \tag{2.10}$$

Also, from equation (2.8), we have

$$ \frac{\partial^m}{\partial x^m}\ {}_HR_n^{(m,s)}(x,y,z;A) = (\sqrt{mA})^m \frac{n!}{(n-m)!}\ {}_HR_{n-m}^{(m,s)}(x,y,z;A) \quad (n \geq m)\ , \tag{2.11}$$

$$ \frac{\partial^s}{\partial x^s}\ {}_HR_n^{(m,s)}(x,y,z;A) = (\sqrt{mA})^s \frac{n!}{(n-s)!}\ {}_HR_{n-s}^{(m,s)}(x,y,z;A) \quad (n \geq s)\ . \tag{2.12}$$

Consequently, we get the following matrix differential relations for the HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$:

$$ \frac{\partial^m}{\partial x^m}\ {}_HR_n^{(m,s)}(x,y,z;A) = -(\sqrt{mA})^m\ \frac{\partial}{\partial y}\ {}_HR_n^{(m,s)}(x,y,z;A), \tag{2.13}$$

$$ \frac{\partial^s}{\partial x^s}\ {}_HR_n^{(m,s)}(x,y,z;A) = (\sqrt{mA})^s\ \frac{\partial}{\partial z}\ {}_HR_n^{(m,s)}(x,y,z;A). \tag{2.14}$$

In view of equations (1.19), (1.22) and (2.1), we have

$$ {}_HR_n^{(m,s)}(x,0,z;A) = R_n^{(s)}(x\sqrt{mA}, z), \tag{2.15}$$

$$ {}_HR_n^{(m,s)}(x,0,0;A) = R_n(x\sqrt{mA}). \tag{2.16}$$

Now, using initial conditions (2.15) and (2.16) in matrix differential relations (2.13) and (2.14), we get the following operational representation for the HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$:

$$ {}_HR_n^{(m,s)}(x,y,z;A) = \exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)\left\{R_n^{(s)}(x\sqrt{mA}, z)\right\}, \tag{2.17}$$

$$ {}_HR_n^{(m,s)}(x,y,z;A) = \exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m} + z(\sqrt{mA})^{-s}\frac{\partial^s}{\partial x^s}\right)\Big\{R_n(x\sqrt{mA})\Big\}. \tag{2.18} $$

A simple computations shows that the operational rule (2.17) can be written in the following form:

$$ {}_HR_n^{(m,s)}(x+w,y,z;A) = \exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)\Big\{R_n^{(s)}((x+w)\sqrt{mA},z)\Big\}. \tag{2.19} $$

In the next section, we frame the HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$ within the context of monomiality principle.

3 Monomiality principle and Hermite-Appell matrix polynomials

In order to frame the HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$ within the context of monomiality principle formalism, we prove the following result:

Theorem 3.1. *The HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$ are quasi-monomial with respect to the following multiplicative and derivative operators*:

$$ \hat{M}_{HA} := x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + sz(\sqrt{mA})^{-(s-1)}\frac{\partial^{s-1}}{\partial x^{s-1}} + \frac{A'(\hat{D}_x/\sqrt{mA})}{A(\hat{D}_x/\sqrt{mA})} \tag{3.1} $$

and

$$ \hat{P} := \frac{1}{\sqrt{mA}}\,\frac{\partial}{\partial x} \tag{3.2} $$

respectively.

Proof Consider the identity

$$ (\hat{D}_x/\sqrt{mA})\Big\{A(t)\exp(xt\sqrt{mA} - yt^mI + zt^sI)\Big\} = t\,\Big\{A(t)\exp(xt\sqrt{mA} - yt^mI + zt^sI)\Big\}. \tag{3.3} $$

Since, $A(t)$ is an invertible series and $\frac{A'(t)}{A(t)}$ has Taylor's series expansion in power of t, therefore, we have

$$ \frac{A'(\hat{D}_x/\sqrt{mA})}{A(\hat{D}_x/\sqrt{mA})}\Big\{A(t)\exp(xt\sqrt{mA} - yt^mI + zt^sI)\Big\} $$

$$ = \frac{A'(t)}{A(t)}\Big\{A(t)\exp(xt\sqrt{mA} - yt^mI + zt^sI)\Big\}, \tag{3.4} $$

where the prime denotes the derivative of the function $A(t)$.

Now, differentiating equation (2.1) partially with respect to t, we have

$$ \left(x\sqrt{mA} - myt^{m-1}I + szt^{s-1}I + \frac{A'(t)}{A(t)}\right)A(t)\exp(xt\sqrt{mA} - yt^mI + zt^sI) $$

$$= \sum_{n=0}^{\infty} {}_HR_{n+1}^{(m,s)}(x,y,z;A)\frac{t^n}{n!}. \tag{3.5}$$

Using equations (3.4) and (2.1) in the l.h.s. of equation (3.5), we find

$$x\sqrt{mA}\sum_{n=0}^{\infty} {}_HR_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!} - my\sum_{n=0}^{\infty} {}_HR_n^{(m,s)}(x,y,z;A)\frac{t^{n+m-1}}{n!}$$
$$+sz\sum_{n=0}^{\infty} {}_HR_n^{(m,s)}(x,y,z;A)\frac{t^{n+s-1}}{n!} + \frac{A'(\hat{D}_x/\sqrt{mA})}{A(\hat{D}_x/\sqrt{mA})}\sum_{n=0}^{\infty} {}_HR_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} {}_HR_{n+1}^{(m,s)}(x,y,z;A)\frac{t^n}{n!}. \tag{3.6}$$

Using equations (2.11) and (2.12) in the l.h.s. of the above equation, we find

$$\Bigg(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + sz(\sqrt{mA})^{-(s-1)}\frac{\partial^{s-1}}{\partial x^{s-1}}$$
$$+\frac{A'(\hat{D}_x/\sqrt{mA})}{A(\hat{D}_x/\sqrt{mA})}\Bigg)\sum_{n=0}^{\infty} {}_HR_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_HR_{n+1}^{(m,s)}(x,y,z;A)\frac{t^n}{n!}. \tag{3.7}$$

Equating the coefficients of like powers of t in both sides of equation (3.7), we get

$$\Bigg(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + sz(\sqrt{mA})^{-(s-1)}\frac{\partial^{s-1}}{\partial x^{s-1}}$$
$$+\frac{A'(\hat{D}_x/\sqrt{mA})}{A(\hat{D}_x/\sqrt{mA})}\Bigg) {}_HR_n^{(m,s)}(x,y,z;A) = {}_HR_{n+1}^{(m,s)}(x,y,z;A), \tag{3.8}$$

which in view of the monomiality principle equation (1.4a) yields assertion (3.1) of Theorem 2.1.

Also, from recurrence relation (2.8) and in view of equations (1.4b), we get assertion (3.2) of Theorem 2.1.

Remark 3.1. Using equations (3.1) and (3.2) in monomiality principle equation (1.6), we get the following matrix differential equation satisfied by the HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$:

$$\Bigg(sz(\sqrt{mA})^{-s}\frac{\partial^s}{\partial x^s} - my(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m} + x\frac{\partial}{\partial x}$$
$$+\frac{1}{\sqrt{mA}}\frac{A'(D_x/\sqrt{mA})}{A(D_x/\sqrt{mA})}\frac{\partial}{\partial x} - n\Bigg) {}_HR_n^{(m,s)}(x,y,z;A) = 0. \tag{3.9}$$

4 Applications

We derive some results for the HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$ from the results of the corresponding 2D Appell polynomials $R_n^{(s)}(x,y)$ by making use of suitable operational rules.

I. Replacing x by $x\sqrt{mA}$ and y by z in equation (1.24) and operating $\exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)$ on the resultant equation and then using equations (1.18) and (2.17) in the l.h.s. and r.h.s. respectively, we get

$$H_k^{(m,s)}(x,y,z;A)=\sum_{k=0}^{n}\binom{n}{k}Q_{n-k}\ {}_HR_k^{(m,s)}(x,y,z;A). \tag{4.1}$$

II. Replacing x by $x\sqrt{mA}$, z by $w\sqrt{mA}$ and y by z in equations (1.25), (1.26) and then operating $\exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)$ on the resultant equations and then using equations (1.18), (2.17) and (2.19), we get

$${}_HR_n^{(m,s)}(x+w,y,z;A)=\sum_{k=0}^{n}\binom{n}{k}{}_HR_k^{(m,s)}(x,y,z;A)(w\sqrt{mA})^{n-k}, \tag{4.2}$$

$${}_HR_n^{(m,s)}(x+w,y,z;A)=\sum_{k=0}^{n}\binom{n}{k}H_k^{(m,s)}(x,y,z;A)A_{n-k}(w\sqrt{mA}), \tag{4.3}$$

respectively.

III. Making the same above replacements in equation (1.25) and operating $\exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial w^m}\right)$ on the resultant equation and then using equations (1.21) and (1.10) in the l.h.s. and r.h.s. respectively, we get

$${}_HR_n^{(m,s)}(x+w,y,z;A)=\sum_{k=0}^{n}\binom{n}{k}R_k^{(s)}(x\sqrt{mA},z)\ H_{n-k,m}(w,y,A). \tag{4.4}$$

IV. Replacing x by $x\sqrt{mA}$, z by $w\sqrt{mA}$, y by z and w by v in equation (1.27) and operating $\exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)$ and $\exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial w^m}\right)$ on the resultant equation respectively and then using equations (2.19), (2.17), (1.18) and (1.15), we get

$${}_HR_n^{(m,s)}(x+w,y,z+v;A)=\sum_{k=0}^{n}\binom{n}{k}{}_HR_k^{(m,s)}(x,y,z;A)H_{n-k,m}^{(s)}(w,v;A) \tag{4.5}$$

and

$${}_HR_n^{(m,s)}(x+w,y,z+v;A)=\sum_{k=0}^{n}\binom{n}{k}R_k^{(s)}(x\sqrt{mA},z)H_{n-k}^{(m,s)}(w,y,v;A), \tag{4.6}$$

respectively.

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